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Assessment of First Order Computational Model for Free Vibration Analysis of FGM Plates

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Abstract— This paper presents the complete theoretical formulation and the analytical solutions for the free vibration analysis of functionally graded material (FGM) plates using First-order Shear Deformation Theory (FSDT). The material properties are assumed to be isotropic along the plane of the plate and vary through the thickness according to the power law function. The equations of motion are obtained using Hamilton's principle. The analytical solutions are obtained in closed-form using Navier's solution technique and by solving the eigenvalue equation.

Index Terms— Analytical solution, Functionally graded Material plates, First-order model, Hamilton's principle, Navier's method, Power law function, Shear deformation.

1 Introduction

THE concept of Functionally Graded Materials (FGMs) was proposed in 1984 by materials scientists as a means of preparing thermal barrier materials. FGMs are the heterogeneous composite materials in which the material properties are gradually varied along certain directions in a predetermined manner. Thus, mitigating the problems induced due to sudden change of thermo-mechanical properties as in the case of laminated composites. FGMs have a great potential of becoming an advanced struc-tural material in various engineering and industrial applications. Therefore, to use them efficiently a good understanding of their structural and dynamical behavior and also an accurate knowledge of the deformation charecteristics, stress distribution, natural frequencies and buckling loads under various load conditions are needed. Several analytical and numerical approaches have been proposed by various authers for the analysis of FGM plates. Hamilton's principle and assumed mode technique were used to study the parametric resonance of FGM rectangular plates based on classical plate theory under harmonic in-plane loading [1]. A meshfree radial point interpolation method was employed for static and dynamic analyses of FGM plates based on First-order Shear Deformation Theory (FSDT) [2]. First five natural frequencies of an FGM plate were maximized using FSDT along with FEM [3]. The CPT was employed to show that FGM plates can be idealized as homogeneous plates by properly selecting the reference surface so that no special tool is required to analyse their behavior [4]. A Levi type solution was employed for free vibration analysis of FGM plates based on FSDT, where two opposite edges are simply supported and other two edges under various boundary conditions [5].

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In this paper, an attempt has been made to compare and assess quantitatively the accuracy of the results obtained using First-order computational model for predicating free vibration response of simply supported FGM plates.

2 Displacement Model

Based on the FSDT the displacement field at a point in an FGM plate is expressed as [7],

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\,\theta_x(x,y), \\ v(x,y,z) &= v_0(x,y) + z\,\theta_y(x,y), \\ w(x,y,z) &= w_0(x,y). \end{aligned} \tag{1}$$

Where the terms u, v and w are the displacements of a general point (x, y, z) in x, y and z directions respectively. Where the terms u_0, v_0 are the in-plane displacements and the term w_0 is the transverse displacement of a general point (x, y) on the middle plane. The functions θ_x, θ_y are rotations of the normal to the middle plane about y and x axes respectively.

The strain is assumed to be linear through the thickness of the FGM plate

$$\varepsilon = \varepsilon^0 + z\varepsilon' \tag{2}$$

$$\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \\ &\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \\ &\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}. \end{aligned} \tag{3}$$

2.1 Constitutive Relationship

Assuming through the thickness gradation of material properties, the volume fraction composition is defined using power law function as,

$$E(z) = E_m + (E_c - E_m)\left(\frac{2z+h}{2h}\right)^p. \tag{4}$$

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Where,

E_m = Young’s modulus of metal,
E_c = Young’s modulus of ceramic,
p=Parameter that dictates the variation of material profile through the thickness,
ν=Poisson’s ratio,
h=Thickness of the plate.

The stress-strain relationship accounting for the transverse shear deformation is given by,

$$\{\sigma\} = [Q]\{\varepsilon\}. \tag{5}$$

Where,

$\{\sigma\}$ = Stress vector,
$[Q]$ = Transformed elastic stiffness matrix,
$\{\varepsilon\}$ = Strain vector.

$$Q_{11} = Q_{22} = \frac{E(z)}{(1-\nu^2)}, \qquad Q_{12} = Q_{21} = \nu Q_{11},$$

$$Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1+\nu)}.$$

2.3 Govering equations

The governing equations of motion are derived using Hamilton’s principle. The equations of motion associated with the present first-order computational model are,

$$\begin{aligned}
\delta u_o &: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1\ddot{u}_0 + I_2\ddot{\theta}_x, \\
\delta v_o &: \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = I_1\ddot{v}_0 + I_2\ddot{\theta}_y, \\
\delta \theta_x &: \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_3\ddot{\theta}_x + I_2\ddot{u}_0, \\
\delta \theta_y &: \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = I_3\ddot{\theta}_y + I_2\ddot{v}_0, \\
\delta w_0 &: \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P_z^+ = I_1\ddot{w}_0.
\end{aligned} \tag{6}$$

Here (N_x, N_y, N_{xy}), (M_x, M_y, M_{xy}) and (Q_x, Q_y) respectively denotes in-plane, bending and shear stress resultants, which can be defined as,

$$\begin{Bmatrix} N_x \\ N_y \\ M_x \\ M_y \end{Bmatrix} = [A]\begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \end{Bmatrix} + [A']\begin{Bmatrix} \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial x} \end{Bmatrix}, \tag{7}$$

$$\begin{Bmatrix} N_{xy} \\ M_{xy} \end{Bmatrix} = [B^{'}]\begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \end{Bmatrix} + [B']\begin{Bmatrix} \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial x} \end{Bmatrix}, \tag{8}$$

$$\begin{Bmatrix} Q_x \\ Q_x^* \end{Bmatrix} = [D]\begin{Bmatrix} \theta_x \\ \frac{\partial w_0}{\partial x} \end{Bmatrix} + [D']\begin{Bmatrix} \theta_y \\ \frac{\partial w_0}{\partial y} \end{Bmatrix}, \tag{9}$$

$$\begin{Bmatrix} Q_y \\ Q_y^* \end{Bmatrix} = [E]\begin{Bmatrix} \theta_y \\ \frac{\partial w_0}{\partial y} \end{Bmatrix} + [E']\begin{Bmatrix} \theta_x \\ \frac{\partial w_0}{\partial x} \end{Bmatrix}. \tag{10}$$

Where the matrices $[A], [A'], [B], [B'], [D], [D'], [E], [E']$ are the matrices of plate stiffness whose elements are defined as,

$$[A] = \int_{-h/2}^{h/2} \begin{bmatrix} Q_{11} & Q_{12} & Q_{11}z & Q_{12}z \\ Q_{12} & Q_{22} & Q_{12}z & Q_{22}z \\ Q_{11}z & Q_{12}z & Q_{11}z^2 & Q_{12}z^2 \\ Q_{12}z & Q_{22}z & Q_{12}z^2 & Q_{22}z^2 \end{bmatrix} dz,$$

$$[B] = \int_{-h/2}^{h/2} \begin{bmatrix} Q_{44} & Q_{44} & Q_{44}z & Q_{44}z \\ Q_{44}z & Q_{44}z & Q_{44}z^2 & Q_{44}z^2 \end{bmatrix} dz,$$

$$[D] = \int_{-h/2}^{h/2} \begin{bmatrix} Q_{66} & Q_{66} \end{bmatrix} dz,$$

$$[E] = \int_{-h/2}^{h/2} \begin{bmatrix} Q_{55} & Q_{55} \end{bmatrix} dz,$$

$$[A'] = [B'] = [C'] = [D'] = 0.$$

2.4 The Navier Solutions

The following boundary conditions are imposed for a simply supported rectangular FGM plate having thickness h with sides a and b.
At edges x=0 and x=a;

$$\begin{aligned} &v_0 = 0;\quad w_0 = 0;\quad \theta_y = 0; \\ &M_x = 0;\quad N_x = 0; \end{aligned} \tag{11}$$

At edges y= 0 and y = b

$$\begin{aligned} &u_0 = 0;\quad w_0 = 0;\quad \theta_x = 0; \\ &M_y = 0;\quad N_y = 0; \end{aligned} \tag{12}$$

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$$u_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{0_{mn}} \cos\alpha x \sin\beta y\, e^{-i\omega t}$$
$$v_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{0_{mn}} \sin\alpha x \cos\beta y\, e^{-i\omega t}$$
$$w_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{0_{mn}} \sin\alpha x \sin\beta y\, e^{-i\omega t} \tag{13}$$
$$\theta_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{x_{mn}} \cos\alpha x \sin\beta y\, e^{-i\omega t}$$
$$\theta_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{y_{mn}} \sin\alpha x \cos\beta y\, e^{-i\omega t}$$

Where,

$$\alpha = \frac{m\pi}{a} \text{ and } \beta = \frac{n\pi}{b}.$$

Where,

$$\alpha = \frac{m\pi}{a} \text{ and } \beta = \frac{n\pi}{b}.$$

Substituting eqn. (11)-(13) in in eqn. (6) and collecting the coefficients, one obtain

$$\left([X]_{5X5} - \lambda[M]_{5X5}\right)\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_x \\ \theta_y \end{Bmatrix}_{5\times 1} = 0, \quad \text{Where } \lambda=\omega^2. \tag{1}$$

For any fixed values of m and n. The elements of coefficient matrix [X] are given as,

$$X_{11} = A_{11}\alpha^2 + B_{11}\beta^2,$$
$$X_{12} = A_{12}\alpha\beta + B_{12}\alpha\beta,$$
$$X_{13} = 0,$$
$$X_{14} = A_{13}\alpha^2 + B_{13}\beta^2,$$
$$X_{15} = A_{14}\alpha\beta + B_{14}\alpha\beta,$$
$$X_{22} = A_{22}\beta^2 + B_{12}\alpha^2,$$
$$X_{23} = 0,$$
$$X_{24} = A_{23}\alpha\beta + B_{13}\alpha\beta,$$
$$X_{25} = A_{24}\beta^2 + B_{14}\alpha^2,$$
$$X_{33} = D_{12}\alpha^2 + E_{12}\beta^2,$$
$$X_{34} = D_{11}\alpha,$$
$$X_{35} = E_{11}\beta,$$
$$X_{44} = A_{33}\alpha^2 + B_{23}\beta^2 + D_{11},$$
$$X_{45} = A_{34}\alpha\beta + B_{24}\alpha\beta,$$
$$X_{55} = A_{44}\beta^2 + B_{24}\alpha^2 + E_{11},$$
$$X_{ij} = X_{ji}, \quad i, j = 1 \text{ to } 5.$$

For any fixed values of m and n. The elements of mass matrix [M] are given as,

$M_{1,1} = I_1$ $M_{1,2} = 0$ $M_{1,3} = 0$ $M_{1,4} = I_2$
$M_{1,5} = 0$ $M_{2,2} = I_1$ $M_{2,3} = 0$ $M_{2,4} = 0$
$M_{2,5} = I_2$ $M_{3,3} = I_1$ $M_{3,4} = 0$ $M_{3,5} = 0$
$M_{4,4} = I_3$ $M_{4,5} = 0$ $M_{5,5} = I_3$

3 Numerical Results and Discussion

In this section, the numerical examples solved are described and discussed. A shear correction factor of 5/6 is used in the present model for computing results. For all the problems, a simply supported rectangular FGM plate with SS-1 boundary conditions is considered for the analysis.

The following sets of data are used in obtaining numerical results, Material set 1 [7].

$E_m = 70\text{GPa}$, $\nu = 0.3$, $\rho = 2702\text{kg/m}^3$,
$E_c = 200\text{GPa}$, $\nu = 0.3$, $\rho = 5700\text{kg/m}^3$,
$p = 1$.

Material set 2 [8].

$E_m = 70\text{GPa}$, $\nu = 0.3$, $\rho = 2702\text{kg/m}^3$,
$E_c = 380\text{GPa}$, $\nu = 0.3$, $\rho = 3800\text{kg/m}^3$,
p = Open.

Results are reported using the following non-dimensional form,

$$\overline{\omega}_{mn} = \omega_{mn}\left(\frac{a^2}{h}\right)\sqrt{\frac{\rho_m}{E_m}}$$

Example 1: A simply supported FGM square plate is considered for the analysis. Material set 1 is used. The nondimensionalized values of natural frequency for various side-to-thickness ratio (a/h) are given in Table 1. The non-dimensionalized values of natural frequency are found to increase with increase in the side-to-thickness ratio. It is found that at lower thickness modes, the present results are in good agreement with the exact three-dimensional elasticity solution [7] where as significant difference between the ptesent results and exact solution exists at higher thickness modes.

Example 2: A simply supported FGM plate with side-to-thickness

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ratio equal to 5 is considered for the analysis. Material set 2 is used. The nondimensionalized values of natural frequency for various aspect ratio (a/b) and power law function are given in Table 2. For any given a/b ratio, the non-dimensionalized natural frequency decreases as the power law function p increases and for any given power law function value p as the a/b value increases, the natural frequency increases.

4 Conclusion

Analytical formulations and solutions to study the free vibration response of simply supported FGM rectangular plates using a first-order computation model are presented. The accuracy of the solution is first established by comparing the results with exact three-dimensional elasticity solution available in the literature. After establishing the accuracy of prediction, new results for the FGM plates with varying side-to-thickness ratio, aspect ratio, and power law function are presented.

TABLE 1
NON-DIMENSIIONALIZED NATURAL FREQUENCY

a/h	Thickness mode	3D exact solution [7]	Present FSDT
5	1	5.4806	5.6908
	2	14.558	15.341
	3	24.381	25.925
	4	53.366	57.812
	5	57.620	62.691
10	1	5.9609	6.1864
	2	29.123	30.686
	3	49.013	51.866
	4	207.50	225.25
	5	212.22	230.60
10	1	6.1076	6.3372
	2	58.250	61.374
	3	98.145	103.74
	4	823.92	894.88
	5	828.78	900.37

TABLE 2
NON-DIMENSIIONALIZED NATURAL FREQUENCY

p	a/b				
	1.0	1.5	2.0	2.5	3.0
0	10.374	15.857	22.683	30.414	38.732
0.5	8.8650	13.602	19.536	26.298	33.614
1	8.0107	12.306	17.700	23.861	30.542
5	6.7767	10.330	14.739	19.720	25.073
10	6.5019	9.8731	14.029	18.695	23.681

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